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A characterization of n -dependent theories

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1 Introduction

The notion of n -dependent property, a generalization of dependent property (NIP), were introduced by Shelah in [1] (2009). Only few basic properties of the n -dependent property are known, although Shelah showed an interesting result on definable groups for 2-dependent theories [2]. In this article, we show a characterization of n -dependent theories by using counting types over finite sets:

Theorem 1. *Let $\varphi(x, y_1, \dots, y_n)$ be an L -formula. φ is n -dependent if and only if there is a constant $\epsilon > 0$ such that $|S_\varphi(\prod_i A_i)| < 2^{k^{n-\epsilon}}$ for sufficiently large $k \in \omega$ and for all $|A_i| = k$.*

Then we see boolean combinations of n -dependent theories are again n -dependent as a corollary of the characterization. This characterization gives a partial answer for a conjecture on the number of types in n -dependent theories by Shelah in [1]:

Conjecture 2 (S. Shelah). Let $\varphi(x, y_1, \dots, y_n)$ be an L -formula and let $m = \text{len}(x)$. φ is n -dependent if and only if $|S_\varphi(\prod_i A_i)| < 2^{mk^{n-1}}$ for all $|A_i| = k$.

(Note that Shelah's conjecture is immediately false where $n = 1$, and you can check that it is also false where $n \neq 1$ with a little discussion. So I think we should replace " mk^{n-1} " by something like " $\beta(\log k)k^{n-1}$ " (β depends on n, φ), to make a sense.)

One of the most important results in this article is a generalization of Sauer-Shelah lemma, a famous combinatorial lemma, discussed in section 3. One will notice that the characterization and (generalized) Sauer-Shelah lemma are two sides of the same coin. This report is a partial result of a study with A. Chernikov and D. Palacin on n -dependent theories.

2 Preliminaries

When we discuss on model theoretic topic, we will use ordinal notation in model theory: $\varphi(x), \psi(y), \dots, M, N, \dots, A, B, \dots$, are used for formulas, models and subsets of models, except x, y, \dots and a, b, \dots are used for tuples of variables and elements, respectively. We work under the big model of a complete L -theory T , so every model and set of elements are contained in it.

When we discuss on combinatorial situation, we will use X, Y, \dots for (universal) sets, V, W, \dots for subsets of X, Y, \dots and v, w, \dots for elements in V, W, \dots .

First of all, we give a definition of n -dependence.

Definition 3. 1. Let $\varphi(x, y_1, \dots, y_n)$ be an L -formula. The formula φ is said to be n -independent if there are sets A_i ($1 \leq i \leq n$) such that for every disjoint subsets X and $Y \subset \prod_i A_i$ there is a tuple b satisfies $\models \bigwedge_{(a_1, \dots, a_n) \in X} \varphi(b, a_1, \dots, a_n) \wedge \bigwedge_{(a_1, \dots, a_n) \in Y} \neg \varphi(b, a_1, \dots, a_n)$. n -dependence is defined by the negation of n -independence.

2. Let T be an L -theory. T is said to be n -dependent (or, have n -dependent property) if every formula $\varphi(x, y_1, \dots, y_n)$ is n -dependent.

Note that $\varphi(x, y)$ is 1-dependent if and only if $\varphi(x, a)$ is independent for some a . It is immediate that n -dependence implies $(n+1)$ -dependence, so n -dependent property is a generalization of NIP .

Definition 4. Let $\varphi(x, y_1, \dots, y_n)$ be an L -formula and A_i ($1 \leq i \leq n$) a set of parameters. Let $B \subset \prod_i A_i$.

1. A φ -types over B is a maximal consistent set of formulas $\varphi(x, a_1, \dots, a_n)$ and $\neg \varphi(x, a_1, \dots, a_n)$ with $(a_1, \dots, a_n) \in B$.
2. $S_\varphi(B)$ is the set of all φ -types over B .

For the proof of the main result, we'll use a graph theoretic fact, as bellow.

Definition 5. Let $n \geq 1$ be a natural number. An n -partite n -hypergraph (V, E) is an n -uniform hypergraph satisfying the following:

- V is a disjoint union of sets V_i ($1 \leq i \leq n$).
- If $E(v_1, \dots, v_n)$ holds then $v_i \in V_i$.

We say (V, E) has size k if $|V_i| = k$ for all i . An n -partite n -hypergraph (V, E) is said to be complete if there is no n -partite n -hypergraph (V, E') with $E' \supsetneq E$. If $n = 1$, the n -hypergraph (V, E) is just a set V and a subset $E \subset V$, and it is complete if $E = V$.

Let G be an n -partite n -hypergraph of size k . If G is complete, then it has k^n edges, and immediately contains copies of complete n -partite n -hypergraphs of size $< k$. The following fact shows that there is ϵ (not depending on the choice of G) such that if G has $k^{n-\epsilon}$ edges then it contains a copy of complete n -partite n -hypergraph of size d .

Fact 6 (Erdős[3]). *Let $d, n > 1$ be natural numbers. Then for sufficiently large $k > n_0$, the following condition holds: Let (V, E) be an n -partite n -hypergraph of size k . If $|E| \geq k^{n-\epsilon}$ with $\epsilon = d^{1-n}$ then (V, E) contains a copy of a complete n -partite n -hypergraph of size d .*

Remark 7. Fact 6 given in [3] doesn't hold where $n = 1$, because $k^{n-\epsilon} = k^0 = 1$. So we replace the lower bound $k^{n-\epsilon}$ by $dk^{n-\epsilon}$, then the fact holds for all $n \geq 1$. This replacement is necessary for our main lemma to include Sauer-Shelah lemma. But it make the inequation in the main lemma more complex.

Our characterization of n -dependent property is related to a combinatorial proposition, called Sauer-Shelah lemma. To explain this lemma, we need to introduce some notions in combinatorics. Most of the following is proved in Hang Q. Ngo's online note [4] and [5].

Definition 8. Let X be a set.

1. A set system \mathcal{H} on X is a subset of the power set $\mathcal{P}(X)$ of X .
2. $\mathcal{H} \cap V := \{W \cap V : W \in \mathcal{H}\}$ for $V \subset X$.
3. We say $V \subset X$ is shuttered by \mathcal{H} if $\mathcal{H} \cap V = \mathcal{P}(V)$.

Definition 9. Let X be an infinite set and \mathcal{H} a set system on X .

1. $\pi_{\mathcal{H}}(k) := \max\{|\mathcal{H} \cap V| : |V| = k\}$. The function $\pi_{\mathcal{H}} : \mathbb{N} \rightarrow \mathbb{N}$ is called a shutter function.
2. VC-dimension (Vapnik-Chervonenkis dimension): $VC(\mathcal{H}) = \max\{k : \pi_{\mathcal{H}}(k) = 2^k\}$.

Fact 10 (Sauer-Shelah lemma). *Let \mathcal{H} be a set system on an infinite set X . Suppose that $VC(\mathcal{H}) = d < \infty$. Then for $n > d$,*

$$\pi_{\mathcal{H}}(k) \leq \sum_{i=1}^d \binom{k}{i} \leq \left(\frac{ek}{d}\right)^d = O(2^{d \log_2(k)}).$$

By using Sauer-Shelah lemma, we have the following:

Fact 11. *Let $\varphi(x, y)$ be an L -formula. $\varphi(x, y)$ is dependent if and only if there is d such that for all $k > d$, $|S_{\varphi}(A)| \leq \left(\frac{ek}{d}\right)^d = O(2^{d \log_2(k)})$, where $|A| = k$.*

One of elegant proofs of Sauer-Shelah lemma is given by Shifting technique, as below.

Fact 12. *Let X be a finite set and \mathcal{H} a set system on X . Then we can find a set system \mathcal{G} on X such that*

- $|\mathcal{H}| = |\mathcal{G}|$,
- if $V \subset X$ is shuttered by \mathcal{G} then V is shuttered by \mathcal{H} ,
- \mathcal{G} is closed under taking subset.

3 A generalization of Sauer-Shelah lemma

In this section, we prove an inequation like Sauer-Shelah lemma. There may be better bound for our inequation, but still it is useful enough to apply to n -dependent theories.

We'll generalize the notions in the previous section to higher dimension. Suppose $n \geq 1$. Let X_i ($1 \leq i \leq n$) be sets of size $m \in \omega \cup \{\omega\}$ and let $X = \prod_i X_i$. Let \mathcal{H} be a set system on X . (Note that $|X| = m^n$, and if X is shuttered by \mathcal{H} then $|\mathcal{H}| = 2^{m^n}$.)

Definition 13. 1. $\pi_{\mathcal{H},n}(k) := \max\{|\mathcal{H} \cap V| : V = \prod_i V_i, V_i \subset X_i, |V_i| = k\}$.

2. VC_n -dimension: $VC_n(\mathcal{H}) = \max\{k : \pi_{\mathcal{H},n}(k) = 2^{k^n}\}$.

Lemma 14 (Main lemma). 1. (precise form) Let $n \geq 1$ and let $VC_n(\mathcal{H}) = d < \infty$. For sufficiently large k , we have

$$\pi_{\mathcal{H},n}(k) \leq \sum_{i=0}^{D(k)} \binom{k^n}{i} \leq \left(\frac{ek^n}{D(k)} \right)^{D(k)} = O(2^{D(k)(\epsilon \log_2 k + \log_2(e/(d+1)))}),$$

where $D(k) = (d+1)k^{n-\epsilon} - 1$ and $\epsilon = (d+1)^{1-n}$. Especially, if $n = 1$ then $\epsilon = 1$ and $D(k) = d$, so we have Sauer-Shelah lemma.

2. (simpler form 1) Let $VC_n(\mathcal{H}) = d < \infty$ and let $\epsilon = (d+1)^{1-n}$. There is β (depends only on d and n) such that $\pi_{\mathcal{H},n}(k) \leq 2^{\beta n^{k-\epsilon} \log k}$ for sufficiently large k .
3. (simpler form 2) Let $VC_n(\mathcal{H}) = d < \infty$. There is ϵ' (depends only on d and n) such that $\pi_{\mathcal{H},n}(k) \leq 2^{k-\epsilon'}$ for sufficiently large k .

Proof. The simpler form is immediately shown from the precise form by taking $\beta > (d+1)\epsilon$ and $\epsilon' < \epsilon$. We'll show the first item. Let $X = \Pi_x X_i$ and \mathcal{H} a set system on X . Let $V_i \subset X_i$ be a set of size k and let $\mathcal{H}_0 = \mathcal{H} \cap V$ with $V = \Pi_i V_i$. We'll check $|\mathcal{H}_0| \leq \sum_{i=0}^{(d+1)k^{n-\epsilon}-1} \binom{k^n}{i}$. By the shifting technique in Fact 12, we can find \mathcal{G} satisfying

- $|\mathcal{H}_0| = |\mathcal{G}|$,
- if $W \subset V$ is shuttered by \mathcal{G} then V is shuttered by \mathcal{H}_0 ,
- \mathcal{G} is closed under taking subset. (Hence if $W \in \mathcal{G}$ then W is shuttered by \mathcal{G} .)

Consider any subset $W \subset V$ with $W = \Pi_i W_i$ and $|W_i| = d+1$. Since $VC_n(\mathcal{H}) = d < \infty$, \mathcal{G} cannot contain W , otherwise W is also shuttered by \mathcal{H}_0 , hence by \mathcal{H} , contradicting to the assumption $VC_n(\mathcal{H}) = d$. Take an element $W' \in \mathcal{G}$. Then we have $W \not\subset W'$ because \mathcal{G} is closed under taking subset. We may regards W' as an n -partite n -hypergraph of size k with vertices $V_1 \sqcup \dots \sqcup V_n$ and edges W' . Then W' has no complete n -partite n -hyper subgraph of size $d+1$. So, by Fact 6 and Remark 7, the number $|W'|$ of edges must be bounded by $(d+1)k^{n-\epsilon}$ where $\epsilon = (d+1)^{1-n}$. Then we have

$$\mathcal{G} \subset \{W' \subset V : |W'| \leq (d+1)k^{n-\epsilon} - 1\},$$

and

$$|\mathcal{G}| \leq |\{W' \subset V : |W'| \leq (d+1)k^{n-\epsilon} - 1\}| \leq \sum_{i=0}^{(d+1)k^{n-\epsilon}-1} \binom{k^n}{i}.$$

The rest of the inequation is shown by a general inequation $\sum_{i=0}^s \binom{t}{i} \leq (et/s)^s$ for $t > s \in \mathbb{N}$.

□

Note that if $VC_n(\mathcal{H}) = \infty$ then $\pi_{\mathcal{H},n}(k) = 2^{k^n}$ for all k . So we have the following dichotomy:

Corollary 15. *Let \mathcal{H} be a set system on $X = \prod_{i=1}^n X_i$ with $|X_i| = \omega$. One of the following holds.*

1. $\pi_{\mathcal{H},n}(k) = 2^{k^n}$ for all k .
2. There is $\epsilon' > 0$ such that for sufficiently large k , $\pi_{\mathcal{H},n}(k) < 2^{k^{n-\epsilon'}}$.

4 Characterizing n -dependent property

First we recall the definition of n -dependent property.

Definition 16. 1. Let $\varphi(x, y_1, \dots, y_n)$ be an L -formula. The formula φ is said to be n -independent if there are sets A_i ($1 \leq i \leq n$) such that for every disjoint subsets X and $Y \subset \prod_i A_i$ there is a tuple b satisfies $\models \bigwedge_{(a_1, \dots, a_n) \in X} \varphi(b, a_1, \dots, a_n) \wedge \bigwedge_{(a_1, \dots, a_n) \in Y} \neg \varphi(b, a_1, \dots, a_n)$. n -dependence is defined by the negation of n -independence.

Let $A = \prod_i A_i$ be a set of parameters with A_i of size k and let $\varphi(x, y_1, \dots, y_n)$ be an L -formula. We want to measure the size of the set $S_\varphi(A)$ of φ -types over A .

Definition 17. Let M be an ω -saturated model of T and let $\varphi(x, y_1, \dots, y_n)$ be an L -formula.

1. For $p \in S_\varphi(\prod_i M^{|y_i|})$, we define $(\Pi M)_p \subset \prod_i M^{|y_i|}$ by $\{(a_1, \dots, a_n) \in \prod_i M^{|y_i|} : \varphi(x, a_1, \dots, a_n) \in p\}$.
2. A set system \mathcal{H}_φ on $\prod_i M^{|y_i|}$ is the set $\{(\Pi M)_p : p \in S_\varphi(\prod_i M^{|y_i|})\}$.

Remark 18. Let $A \subset \Pi_i M^{|y_i|}$. The following are immediate from the definitions.

1. $M_p \cap A = M_q \cap A$ if and only if $p|A = q|A$.
2. $|\mathcal{H}_\varphi \cap A| = |S_\varphi(A)|$.
3. φ is n -dependent if and only if $VC_n(\mathcal{H}) = d < \infty$.

With the above remark, we can calculate the number of types by counting $\mathcal{H}_\varphi \cap A$.

Theorem 19. *Let $\varphi(x, y_1, \dots, y_n)$ be an L -formula. The following are equivalent.*

1. φ is n -dependent.
2. For sufficiently large k , if $A = \Pi_i A_i$ with $|A_i| = k$, then $|S_\varphi(A)| \leq \sum_{i=0}^{D(k)} \binom{k^n}{i} \leq \left(\frac{ek^n}{D(k)} \right)^{D(k)} = O(2^{D(k)(\epsilon \log_2 k + \log_2(e/(d+1)))})$, where $D(k) = (d+1)k^{n-\epsilon} - 1$ and $\epsilon = (d+1)^{1-n}$. Especially, the case $n = 1$ implies the well know characterization of dependent property.
3. Let $\epsilon = (d+1)^{1-n}$. There is β such that for sufficiently large k , $|S_\varphi(A)| \leq 2^{\beta k^{n-\epsilon} \log_2 k}$ for all $A = \Pi_i A_i$ with $|A_i| = k$.
4. There is ϵ' such that for sufficiently large k , $|S_\varphi(A)| \leq 2^{k^{n-\epsilon'}}$ for all all $A = \Pi_i A_i$ with $|A_i| = k$.

Proof. Immediately shown from Lemma 14 and Remark 18. □

Corollary 20. *n -dependent formulas are closed under taking boolean combinations.*

Proof. Let $\varphi(x, y_1, \dots, y_n)$ and $\psi(x, y_1, \dots, y_n)$ be n -dependent formulas. By the definition, the negation of n -dependent formula is n -dependent. On the other hand, $|S_{\varphi \wedge \psi}(A)| \leq |S_\varphi(A)| \times |S_\psi(A)| \leq 2^{k^{n-\epsilon'}} \times 2^{k^{n-\epsilon''}} \leq 2^{k^{n-\epsilon'''}}$ for some ϵ', ϵ'' and ϵ''' . So $\varphi \wedge \psi$ is n -dependent. □

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